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COMPLETE ORTHOGONAL SYSTEMS OF 3D SPHEROIDAL MONOGENICS

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Abstract. *In this paper we review two distinct complete orthogonal systems of monogenic polynomials over 3D prolate spheroids. The underlying functions take on either values in the reduced and full quaternions (identified, respectively, with $\mathbb{R}^3$ and $\mathbb{R}^4$), and are generally assumed to be nullsolutions of the well known Riesz and Moisil Théodoresco systems in $\mathbb{R}^3$. This will be done in the spaces of square integrable functions over $\mathbb{R}$ and $\mathbb{H}$. The representations of these polynomials are explicitly given. Additionally, we show that these polynomial functions play an important role in defining the Szegö kernel function over the surface of 3D spheroids. As a concrete application, we prove the explicit expression of the monogenic Szegö kernel function over 3D prolate spheroids.*

1 INTRODUCTION

Quaternion analysis is thought to generalize onto the multidimensional situation the classical theory of holomorphic functions of one complex variable, and to provide the foundations for a refinement of classical harmonic analysis. The rich structure of this function theory involves the analysis of monogenic functions defined in open subsets of $\mathbb{R}^3$, which are nullsolutions of higher-dimensional Cauchy-Riemann systems. In this paper we review two distinct complete orthogonal systems of monogenic polynomials over 3D prolate spheroids. We show that these polynomial functions play an important role in defining the monogenic Szegö kernel function over 3D spheroids. The underlying spheroidal prolate functions (C. Flammer [12], E.W. Hobson [21], N.N. Lebedev [24]) were introduced by C. Niven in 1880 while studying the conduction of heat in an ellipsoid of revolution, which lead to a Helmholtz equation in spheroidal coordinates. The prolate spheroidal harmonics are special functions in mathematical physics which have found many important practical applications in science and engineering where the spheroidal coordinate system is used. They usually appear in the solutions of Dirichlet problems in spheroidal domains arising in hydrodynamics, elasticity and electromagnetism. For the solvability of boundary value problems of radiation, scattering, and propagation of acoustic signals and electromagnetism waves in spheroidal structures, spheroidal wave functions are commonly encountered. Recently, there has been a growing interest in developing numerical methods using prolate spheroidal functions as basis functions [2, 3, 41, 42, 43]. These applications have stimulated a surge of new techniques and have reawakened interest in approximation theory, potential theory, and the theory of partial differential equations of elliptic type for spheroidal domains. Higher dimensional extensions of the prolate spheroidal functions were first studied by Slepian in [35], which provided many of their analytical properties, as well as properties that support the construction of numerical schemes (see also A.I. Zayed [44]). Very recently, K.I. Kou et al. [23] introduced the continuous Clifford prolate spheroidal functions in the finite Clifford Fourier transform setting. These generalized spheroidal functions (for offset Clifford linear canonical transform) were successfully applied for the analysis of the energy concentration problem introduced in the early-sixties by D. Slepian and H.O. Pollak [34].

Since the foundations of the theory of approximation of monogenic functions by Fueter [13, 14], the study of orthogonal polynomials in application to certain boundary value problems for elliptic partial differential equations has been of great importance in connection with certain problems of mathematical physics. In our view much of the older theory has progressed considerably upon the study of monogenic polynomial approximations in the context of quaternion analysis. For a detailed historic survey and extended list of references on monogenic approximations we refer to [17]. Most relevant to our study are the intimate connections between monogenic functions and spheroidal structures, and the potential flexibility afforded by a spheroid's non-spherical canonical geometry. Developments are described in the sequence of papers by H. Malonek et al. in [1, 26, 27] (cf. [9]) and J. Morais et al. in [18, 19, 20, 30]. In light of this, in [29, 31] (cf. [16]) a very recent approach has been developed to discuss approximation properties for monogenic functions over 3D prolate spheroids by Fourier expansions in monogenic polynomials of which could be explicitly expressed in terms of products of Ferrer's associated Legendre functions multiplied by Chebyshev polynomial factors (see Theorem 3.1 below). Within the scope of this paper we shall be fully concerned with the polynomials introduced in these notes. Studies have shown that the underlying spheroidal monogenics play an important role in defining the monogenic Szegö kernel function for 3D spheroids [32].

2 PRELIMINARIES

2.1 The Riesz and Moisil-Théodoresco systems

As is well known, a holomorphic function $f(z) = u(x, y) + iv(x, y)$ defined in an open domain of the complex plane, satisfies the Cauchy-Riemann system

$$\begin{cases} \dfrac{\partial u}{\partial x} &= \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} &= -\dfrac{\partial v}{\partial x} \end{cases}.$$

As in the case of two variables, we may now characterize two possible analogues of the Cauchy-Riemann system in an open domain of the Euclidean space $\mathbb{R}^3$. More precisely, consider the pair $f = (f_0, f^*)$ where f_0 is a real-valued continuously differentiable function defined on an open domain $\Omega \subset \mathbb{R}^3$ and $f^* = (f_1, f_2, f_3)$ is a continuously differentiable vector-field in Ω for which

$$(\mathrm{R}) \quad \begin{cases} \operatorname{div} f^* &= 0 \\ \operatorname{rot} f^* &= 0 \end{cases}. \tag{2.1}$$

Recall that the 3-tuple f^* is said to be an *M. Riesz system of conjugate harmonic functions* in the sense of E.M. Stein and G. Weiß [36, 37], and system (R) is called the *Riesz system* [33]. The Riesz system has a physical relevance as it describes the velocity field of a stationary flow of a non-compressible fluid without sources nor sinks.

The *Moisil-Théodoresco system* is represented by [28] (cf. [22])

$$(\mathrm{MT}) \quad \begin{cases} \operatorname{div} f^* &= 0 \\ \operatorname{grad} f_0 + \operatorname{rot} f^* &= 0 \end{cases}, \tag{2.2}$$

and it is closely related to many mathematical models of relevance in spatial physical problems such as the Lamé and Stokes systems. Both systems are historical precursors that generalize the classical Cauchy-Riemann system in the plane. Obviously (2.1) may be derived from (2.2) by taking $f_0 = 0$.

2.2 Quaternion analysis

To start with, the (R)- and (MT)-systems may be obtained consistently by working with the quaternion algebra. Let $\mathbb{H} := \{\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} : \ z_l \in \mathbb{R}, \ l = 0, 1, 2, 3\}$ be the Hamiltonian skew field, where the imaginary units **i**, **j**, and **k** are subject to the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1;$$
$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \qquad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \qquad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The scalar and vector parts of $\mathbf{z}$, $\mathbf{Sc}(\mathbf{z})$ and $\mathbf{Vec}(\mathbf{z})$, are defined as the z_0 and $z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$ terms, respectively. Like in the complex case, the conjugate of $\mathbf{z}$ is the reduced quaternion $\overline{\mathbf{z}} = z_0 - z_1\mathbf{i} - z_2\mathbf{j} - z_3\mathbf{k}$, and the norm $|\mathbf{z}|$ of $\mathbf{z}$ is defined by $|\mathbf{z}| = \sqrt{\mathbf{z}\overline{\mathbf{z}}} = \sqrt{\overline{\mathbf{z}}\mathbf{z}} = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}$.

The *paravector space* is the linear subspace defined by $\mathcal{A} := \text{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H}$, with elements of the form $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j}$. Of course, it is assumed here that $\mathcal{A}$ is a real vectorial subspace, but not a subalgebra of $\mathbb{H}$. Now, let Ω be an open subset of $\mathbb{R}^3$ with a piecewise smooth boundary. We say that

$$\mathbf{f} : \Omega \longrightarrow \mathbb{H}, \qquad \mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j} + [\mathbf{f}(x)]_3\mathbf{k} \tag{2.3}$$

is a quaternion-valued function or, briefly, an $\mathbb{H}$-valued function, where the components $[\mathbf{f}]_l$ ($l = 0, 1, 2, 3$) are real-valued functions defined in Ω. By now, it is clear that the form of a paravector-valued function may be derived from (2.3) by taking $[\mathbf{f}(x)]_3 = 0$. Continuity, differentiability, integrability, and so on, which are ascribed to $\mathbf{f}$ are defined componentwise. We will work with both the real- (resp. quaternionic-) linear Hilbert space of square integrable $\mathcal{A}$- (resp. $\mathbb{H}$-) valued functions defined in Ω, that we denote by $L^2(\Omega; \mathcal{A}; \mathbb{R})$ (resp. $L^2(\Omega; \mathbb{H}; \mathbb{H})$). In this assignment, the scalar and quaternionic inner products are defined by

$$< \mathbf{f}, \mathbf{g} >_{L^2(\Omega;\mathcal{A};\mathbb{R})} = \int_\Omega \mathbf{Sc}(\overline{\mathbf{f}}\,\mathbf{g})\, dV \tag{2.4}$$

and

$$< \mathbf{f}, \mathbf{g} >_{L^2(\Omega;\mathbb{H};\mathbb{H})} = \int_\Omega \overline{\mathbf{f}}\,\mathbf{g}\, dV\,,$$

where dV denotes the Lebesgue measure on Ω. For continuously real-differentiable $\mathcal{A}$-valued functions $\mathbf{f}$, the reader may be familiar with the (reduced) quaternionic operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2},$$

which is called generalized Cauchy-Riemann operator on $\mathbb{R}^3$. From this operator we obtain the usual Dirac operator

$$\partial = \mathbf{i}\frac{\partial}{\partial y_1} + \mathbf{j}\frac{\partial}{\partial y_2} + \mathbf{k}\frac{\partial}{\partial y_3}$$

via the equality $\partial = -\mathbf{j}D\mathbf{i}$, and the identification

$$\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathcal{A} \quad \rightarrow \quad \mathbf{y} = x_2\mathbf{i} + x_1\mathbf{j} + x_0\mathbf{k} \in \mathbb{H}.$$

Namely, a continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is said to be *monogenic* in Ω if $D\mathbf{f} = 0 = \mathbf{f}D$ in Ω, which is equivalent to the Riesz system

$$\text{(R)} \quad \begin{cases} \dfrac{\partial[\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial[\mathbf{f}]_1}{\partial x_1} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_2} = 0\,, \\[2ex] \dfrac{\partial[\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_1}{\partial x_0} = 0, \quad \dfrac{\partial[\mathbf{f}]_0}{\partial x_2} + \dfrac{\partial[\mathbf{f}]_2}{\partial x_0} = 0, \quad \dfrac{\partial[\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_1} = 0\,. \end{cases}$$

This system can also be written in abbreviated form:

$$\begin{cases} \operatorname{div} \overline{\mathbf{f}} &= 0 \\ \operatorname{curl} \overline{\mathbf{f}} &= 0 \end{cases}.$$

For the interpretation of the (R)-system in viewpoint of $\mathbb{H} \cong \mathcal{C}\ell_{0,3}^{+}$ we refer to [10]. Following [25], the solutions of the system (R) are customary called (R)-solutions. The subspace of polynomial (R)-solutions of degree n will be denoted by $\mathcal{R}^{+}(\Omega; \mathcal{A}; n)$. We also denote by $\mathcal{R}^{+}(\Omega; \mathcal{A}) := L^2(\Omega; \mathcal{A}; \mathbb{R}) \cap \ker D$ the space of square integrable $\mathcal{A}$-valued monogenic functions defined in Ω.

The analysis of functions with values in $\mathbb{H}$ requires a different treatment. Namely, an $\mathbb{H}$-valued function $\mathbf{f}$ is called *left* (resp. *right*) monogenic in Ω if $\mathbf{f}$ is in $C^1(\Omega; \mathbb{H})$ and satisfies $\partial \mathbf{f} = 0$ (resp. $\mathbf{f}\partial = 0$) in Ω. Throughout the text we only use left $\mathbb{H}$-valued monogenic functions that, for simplicity, we call monogenic. Nevertheless, all results accomplished to left $\mathbb{H}$-valued monogenic functions can be easily adapted to right $\mathbb{H}$-valued monogenic functions. For any $\mathbb{H}$-valued function $\mathbf{f}$ it is worthy of note that the equation $\partial \mathbf{f} = 0$ is equivalent to the system

$$
(\mathrm{MT}) \quad \left\{ \begin{array}{rcl}
\dfrac{\partial [\mathbf{f}]_1}{\partial x_0} + \dfrac{\partial [\mathbf{f}]_2}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_3}{\partial x_2} & = & 0 \\
\dfrac{\partial [\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial [\mathbf{f}]_2}{\partial x_2} + \dfrac{\partial [\mathbf{f}]_3}{\partial x_1} & = & 0 \\
\dfrac{\partial [\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial [\mathbf{f}]_3}{\partial x_0} & = & 0 \\
\dfrac{\partial [\mathbf{f}]_0}{\partial x_2} - \dfrac{\partial [\mathbf{f}]_1}{\partial x_1} + \dfrac{\partial [\mathbf{f}]_2}{\partial x_0} & = & 0
\end{array} \right.
$$

or, in a more compact form:

$$
\left\{ \begin{array}{rcl}
\mathrm{div}\,(\mathbf{Vec}(\mathbf{f})) & = & 0 \\
\mathrm{grad}\,[\mathbf{f}]_0 + \mathrm{rot}\,(\mathbf{Vec}(\mathbf{f})) & = & 0.
\end{array} \right.
$$

For the interpretation of the (MT) system in viewpoint of $\mathbb{H} \cong \mathcal{C}\ell_{0,3}^{+}$ we also refer to [11]. To state our general results we shall need some further notation. The solutions of the (MT)-system are called (MT)-solutions, and the subspace of polynomial (MT)-solutions of degree n is denoted by $\mathcal{M}^{+}(\Omega; \mathbb{H}; n)$. In [38], A. Sudbery proved that $\dim \mathcal{M}^{+}(\Omega; \mathbb{H}; n) = n + 1$. We also denote by $\mathcal{M}^{+}(\Omega; \mathbb{H}) := L^2(\Omega; \mathbb{H}; \mathbb{H}) \cap \ker \partial$ the space of square integrable $\mathbb{H}$-valued monogenic functions defined in Ω.

3 COMPLETE ORTHOGONAL SYSTEMS OF MONOGENIC POLYNOMIALS OVER 3D PROLATE SPHEROIDS

3.1 Prolate spheroidal monogenics

A prolate spheroid is generated by rotating an ellipse about its major axis. For the prolate spheroidal coordinate system (μ, θ, ϕ) the coordinate surfaces are two families of orthogonal surfaces of revolution. The surfaces of constant μ are a family of confocal prolate spheroids, and the surfaces of constant θ are a family of confocal hyperboloids of revolution.

In prolate spheroidal coordinates (see e.g. E.W. Hobson [21], N.N. Lebedev [24]), the Cartesian coordinates may be parameterized by $x = x(\mu, \theta, \phi)$, $\mu \in [0, \infty)$, $\theta \in [0, \pi)$, and $\phi \in [0, 2\pi)$, such that

$$
x_0 = ca\cos\theta, \quad x_1 = cb\sin\theta\cos\phi, \quad x_2 = cb\sin\theta\sin\phi,
$$

where c is the prolatness parameter, and $a = \cosh\mu$, $b = \sinh\mu$, are respectively, the semimajor and semiminor axis of the generating ellipse. Using these transformation relations the surfaces of revolution for which μ is the parameter consist of the confocal prolate spheroids:

$$\mathcal{S} : \frac{x_0^2}{c^2\cosh^2\mu} + \frac{x_1^2 + x_2^2}{c^2\sinh^2\mu} = 1. \tag{3.1}$$

Accordingly, the surface of $\mathcal{S}$ is matched with the surface of the supporting spheroid $\mu = \alpha$ if we put $c^2\cosh^2\alpha = a^2$, and $c^2\sinh^2\alpha = b^2$. Then we obtain the prolatness parameter $c = \sqrt{a^2 - b^2} \in (0,1)$, which means that c is the eccentricity of the ellipse with foci on the x_0-axis: $(-c,0,0)$, $(+c,0,0)$.

In [29] J. Morais found it necessary to focus the discussion on spaces of square integrable functions over $\mathbb{R}$. With this outcome in mind, a complete orthogonal set

$$\{\boldsymbol{\mathcal{E}}_{n,l}, \boldsymbol{\mathcal{F}}_{n,m} : l = 0,\ldots,n+1,\ m = 1,\ldots,n+1\}$$

of polynomial nullsolutions of the well known Riesz system has been developed over 3D prolate spheroids. The mentioned spheroidal monogenics are explicitly given by[1]

Theorem 3.1. *Monogenic polynomials of the form*

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_{n,l}(\mu,\theta,\phi) \;:=\; & \frac{(n+l+1)}{2} A_{n,l}(\mu,\theta)\, T_l(\cos\phi) \\
+ \; & \frac{1}{4\,(n-l+1)} A_{n,l+1}(\mu,\theta)\, [T_{l+1}(\cos\phi)\mathbf{i} + \sin\phi\, U_l(\cos\phi)\mathbf{j}] \\
+ \; & \frac{1}{4}(n+1+l)(n+l)(n-l+2) A_{n,l-1}(\mu,\theta)\, [-T_{l-1}(\cos\phi)\mathbf{i} + \sin\phi\, U_{l-2}(\cos\phi)\mathbf{j}],
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\mathcal{F}}_{n,m}(\mu,\theta,\phi) \;:=\; & \frac{(n+m+1)}{2} A_{n,m}(\mu,\theta) \sin\phi\, U_{m-1}(\cos\phi) \\
+ \; & \frac{1}{4\,(n-m+1)} A_{n,m+1}(\mu,\theta)\, [\sin\phi\, U_m(\cos\phi)\mathbf{i} - T_{m+1}(\cos\phi)\mathbf{j}] \\
- \; & \frac{1}{4}(n+1+m)(n+m)(n-m+2) A_{n,m-1}(\mu,\theta)\, [\sin\phi\, U_{m-2}(\cos\phi)\mathbf{i} + T_{m-1}(\cos\phi)\mathbf{j}],
\end{aligned}$$

for $l = 0,\ldots,n+1$ *and* $m = 1,\ldots,n+1$, *with the notation*

$$A_{n,l}(\mu,\theta) := \sum_{k=0}^{\lceil \frac{n-l}{2} \rceil} \frac{(2n+1-2k)\,(n+l)_{2k}}{(n+1-l)_{2k+1}}\, P^l_{n-2k}(\cosh\mu)\, P^l_{n-2k}(\cos\theta) \tag{3.2}$$

such that $A_{n,-1} = -\frac{1}{n(n+1)^2(n+2)} A_{n,1}$ *form a complete orthogonal system for the interior of the prolate spheroid (3.1) in the sense of the scalar product (2.4). Here* P^l_n *denotes the Ferrer's associated Legendre functions of degree* n *and order* l *of the first kind,* T_l *and* U_l *are the Chebyshev polynomials of the first and second kinds, respectively. Also, we set* $P_n(\cosh\mu) = P^0_n(\cosh\mu)$ *and* $P^l_n(\cosh\mu) = (-1)^l(\sinh\mu)^l \frac{d^l}{dt^l}[P_n(t)]\Big|_{t=\cosh\mu}$.

[1] The first author wishes to thank Mr. N.M. Hung, who has found a misprint in the expressions of the polynomials introduced in [29], and who has shown great interest in questions related to them.

We shall now be able to extend these results to a quaternionic Hilbert subspace; in particular, we exploit a complete orthogonal system of polynomial nullsolutions of the Moisil-Théodoresco system over 3D prolate spheroids. In continuation of [29] (cf. [16]) we designate the new $n+1$ (prolate) spheroidal monogenics by

$$\boldsymbol{\mathcal{S}}_{n,l} := \boldsymbol{\mathcal{E}}_{n,l+1}\,\mathbf{i} + \boldsymbol{\mathcal{F}}_{n,l+1}\,\mathbf{j}, \quad l = 0, \ldots, n, \tag{3.3}$$

namely functions with respect to the variables μ, θ, and the azimuthal angle ϕ of the quaternion form:

$$\begin{aligned}
\boldsymbol{\mathcal{S}}_{n,l}\,(\mu,\theta,\phi) \;&:=\; \frac{1}{2}\,(n+2+l)\,(n+1+l)\,(n-l+1)\,A_{n,l}(\mu,\theta)\,T_l(\cos\phi) \\
&+\; \frac{1}{2}\,(n+2+l)\,A_{n,l+1}(\mu,\theta)\,T_{l+1}(\cos\phi)\,\mathbf{i} \\
&+\; \frac{1}{2}\,(n+2+l)\,A_{n,l+1}(\mu,\theta)\,\sin\phi\,U_l(\cos\phi)\,\mathbf{j} \\
&-\; \frac{1}{2}\,(n+2+l)\,(n+1+l)\,(n-l+1)\,A_{n,l}(\mu,\theta)\,\sin\phi\,U_{l-1}(\cos\phi)\,\mathbf{k}
\end{aligned}$$

with the subscript coefficient function $A_{n,l}(\mu,\theta)$ given by (3.2). It is easily verified that the polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ are the zero functions for $l \geq n+1$.

Remark 3.2. For the usual applications we define these $n+1$ polynomials in a spheroid which has an infinite boundary, because $P_n^l(\cosh\mu)$ becomes infinite with μ. Of course, the results can be extended to the case of the region outside a spheroid as well. One has merely to replace the Ferrer's associated Legendre functions by the Legendre functions of second kind [21].

These $n+1$ polynomials satisfy the first order partial differential equation

$$\begin{aligned}
0 \;&=\; c\,\partial\boldsymbol{\mathcal{S}}_{n,l} \\
&=\; \mathbf{i}\left(\frac{\cos\theta\sinh\mu}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} - \frac{\sin\theta\cosh\mu}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta}\right) \\
&+\; \mathbf{j}\left(\frac{\sin\theta\cosh\mu\cos\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \frac{\cos\theta\sinh\mu\cos\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta} - \frac{\sin\phi}{\sin\theta\sinh\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi}\right) \\
&+\; \mathbf{k}\left(\frac{\sin\theta\cosh\mu\sin\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \frac{\cos\theta\sinh\mu\sin\phi}{\sin^2\theta+\sinh^2\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta} + \frac{\cos\phi}{\sin\theta\sinh\mu}\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi}\right).
\end{aligned}$$

We further assume the reader to be familiar with the fact that ∂ is a square root of the Laplace operator in $\mathbb{R}^3$ in the sense that

$$\begin{aligned}
\Delta_3\boldsymbol{\mathcal{S}}_{n,l} \;&=\; -\partial^2\boldsymbol{\mathcal{S}}_{n,l} \\
&=\; \frac{1}{c^2(\sin^2\theta+\sinh^2\mu)}\left(\frac{\partial^2\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu^2} + \frac{\partial^2\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta^2} + \coth\mu\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\mu} + \cot\theta\frac{\partial\boldsymbol{\mathcal{S}}_{n,l}}{\partial\theta}\right) \\
&+\; \frac{1}{c^2\sin^2\theta\sinh^2\mu}\frac{\partial^2\boldsymbol{\mathcal{S}}_{n,l}}{\partial\phi^2}.
\end{aligned}$$

Remark 3.3. It is of interest to remark at this point that the Laplacian in (prolate) spheroidal coordinates reduces to the classical Laplacian in spherical coordinates if $a = b$, which occurs as μ appoaches infinity, and in which case the two foci coincide at the origin.

In [31] it is shown that the above-mentioned polynomails are (MT)-solutions and form a complete orthogonal system over the interior of 3D prolate spheroids. The principal point of interest is that the orthogonality of the polynomials in question does not depend on the shape of the spheroids, but only on the location of the foci of the ellipse generating the spheroid. It is shown a corresponding orthogonality over the surface of these spheroids with respect to a suitable weight function.

Theorem 3.4 (see [31]). *The monogenic polynomials $\boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\phi)$ $(l=0,\dots,n)$ form a complete orthogonal system over the surface of a prolate spheroid in the sense of the product*

$$\int_{\partial\mathcal{S}} \overline{\mathbf{f}}\,\mathbf{g}\,\boldsymbol{\omega}\,d\sigma\,, \tag{3.4}$$

where $\partial\mathcal{S}$ is the surface of $\mathcal{S}$ and $d\sigma$ denotes the area element on $\partial\mathcal{S}$, and with weight function

$$\boldsymbol{\omega} := |c^2-(ca\cos\theta+\mathbf{i}\,cb\sin\theta)^2|^{1/2}(\sin^2\theta+\sinh^2\mu) \quad (a>b)$$

equal to the square root of the product of the distances from any point inside of the spheroid to the points $(c,0,0)$ and $(-c,0,0)$, and their norms are given by

$$\begin{aligned}
\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\partial\mathcal{S};\mathbb{H};\mathbb{H})} &= \pi\,(n+2+l)\,\frac{(n+2+l)!}{(n-l)!}\\
&\times \Big[(n+1+l)(n-l+1)\,P_n^l(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P_{n+1}^l(\cosh\alpha)\\
&\qquad -\frac{(n+1+l)^2(n-l+1)}{2n+3}\,[P_n^l(\cosh\alpha)]^2\,\sinh\alpha\\
&\qquad +P_n^{l+1}(\cosh\alpha)\,\sinh\alpha\,\cosh\alpha\,P_{n+1}^{l+1}(\cosh\alpha)\\
&\qquad -\frac{(n+2+l)}{2n+3}\,[P_n^{l+1}(\cosh\alpha)]^2\,\sinh\alpha\Big].
\end{aligned}$$

3.2 Properties

This subsection summarizes some basic properties of the prolate spheroidal monogenics.

Proposition 3.5 (see [32]). *The monogenic polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ $(l=0,\dots,n)$ satisfy the following properties:*

1. $\boldsymbol{\mathcal{S}}_{n,l}(0,0,0)=\begin{cases}\frac{(n+2)(n+1)^2}{2}\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\frac{(2n+1-2k)\,(n)_{2k}}{(n+1)_{2k+1}} &, \quad l=0\\ 0 &, \quad l>0\end{cases};$

2. $\boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\pi)=\frac{(n+2+l)}{2}(-1)^l\Big[(n+1+l)(n-l+1)A_{n,l}(\mu,\theta)-A_{n,l+1}(\mu,\theta)\mathbf{i}\Big]$;

3. $\lim_{\phi\longrightarrow 2\pi}\boldsymbol{\mathcal{S}}_{n,l}(\mu,\theta,\phi)=\frac{(n+2+l)}{2}\Big[(n+1+l)(n-l+1)A_{n,l}(\mu,\theta)+A_{n,l+1}(\mu,\theta)\mathbf{i}\Big]$;

4. $\boldsymbol{\mathcal{S}}_{n,1}(\mu,\theta,\phi)=\frac{(n+3)}{2}\Big[n(n+2)A_{n,1}(\mu,\theta)+A_{n,2}(\mu,\theta)\mathbf{i}e^{-\mathbf{k}\phi}\Big]e^{-\mathbf{k}\phi}$;

5. *The polynomials $\boldsymbol{\mathcal{S}}_{n,l}$ are 2π-periodic with respect to the variable ϕ.*

3.3 Numerical examples

This subsection presents some numerical examples showing approximations up to degree 10 for the image of a prolate spheroid under a special spheroidal monogenic mapping. To begin with, a direct observation shows that for each degree $n \in \mathbb{N}_0$ the polynomial $\boldsymbol{\mathcal{S}}_{n,0}$ is monogenic from both sides ($\partial \boldsymbol{\mathcal{S}}_{n,0} = \boldsymbol{\mathcal{S}}_{n,0} \partial = 0$) and is such that $[\boldsymbol{\mathcal{S}}_{n,0}]_3 = 0$, i.e. $\boldsymbol{\mathcal{S}}_{n,0} : \mathcal{S} \longrightarrow \mathcal{A}$. We use this insight to motivate our numerical procedures for computing the image of a 3D prolate spheroid under $\boldsymbol{\mathcal{S}}_{n,0}$. We did not go further than $n = 10$, as our program becomes very time-consuming. Figures $1-3$ visualize approximations of degrees 3, 7 and 10 for the image of a prolate spheroid with semi-axes $a = 4$ and $b = \sqrt{15}$, and centered at the origin.

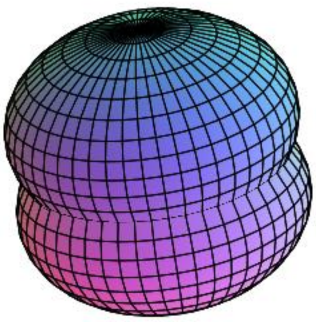

Fig. 1:

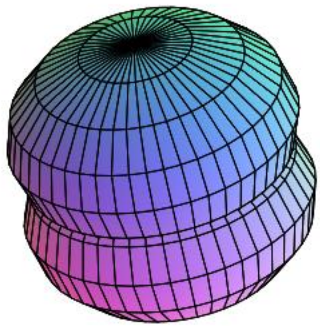

Fig. 2:

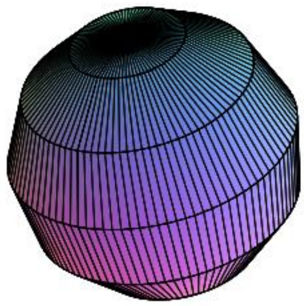

Fig. 3:

3.4 A special Fourier expansion by means of spheroidal monogenics

This subsection discusses a suitable Fourier expansion for monogenic functions over 3D prolate spheroids in terms of orthogonal monogenic polynomials. To begin with, note that for each degree $n \in \mathbb{N}_0$ the set

$$\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0, \ldots, n\} \tag{3.5}$$

is formed by $n + 1 = \dim \mathcal{M}^+(\mathcal{S}; \mathbb{H}; n)$ monogenic polynomials, and therefore, it is complete in $\mathcal{M}^+(\mathcal{S}; \mathbb{H}; n)$. Furthermore, based on the orthogonal decomposition

$$\mathcal{M}^+(\mathcal{S}; \mathbb{H}) = \oplus_{n=0}^{\infty} \mathcal{M}^+(\mathcal{S}; \mathbb{H}; n),$$

and the completeness of the system in each subspace $\mathcal{M}^+(\mathcal{S}; \mathbb{H}; n)$, it follows the result.

Theorem 3.6. *For each n, the set (3.5) forms an orthogonal basis in the subspace $\mathcal{M}^+(\mathcal{S}; \mathbb{H}; n)$ in the sense of the product (3.4) with weight function*

$$\boldsymbol{\omega} := |c^2 - (ca \cos\theta + \mathbf{i}\, cb \sin\theta)^2|^{1/2} (\sin^2\theta + \sinh^2\mu)$$

such that $a > b$. Consequently,

$$\{\boldsymbol{\mathcal{S}}_{n,l} : l = 0, \ldots, n;\ n = 0, 1, \ldots\} \tag{3.6}$$

is an orthogonal basis in $\mathcal{M}^+(\mathcal{S}; \mathbb{H})$.

From now on we shall denote by $\boldsymbol{\mathcal{S}}^*_{n,l}$ $(l = 0, \ldots, n)$ the new normalized basis functions $\boldsymbol{\mathcal{S}}_{n,l}$ in $L^2(\mathcal{S}; \mathbb{H}; \mathbb{H})$ endowed with the inner product (3.4). Yet clearly we can easily write down the Fourier expansion of a square integrable $\mathbb{H}$-valued monogenic function over prolate spheroids in $\mathbb{R}^3$. Next we formulate the result.

Lemma 3.7. *Let* $\mathbf{f} \in \mathcal{M}^+(\mathcal{S}; \mathbb{H})$. *The function* $\mathbf{f}$ *can be uniquely represented with the orthogonal system (3.6):*

$$\mathbf{f}(x) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \boldsymbol{\mathcal{S}}^*_{n,l}\, a^*_{n,l}\,, \tag{3.7}$$

where for each $n \in \mathbb{N}_0$, *the associated (quaternion-valued) Fourier coefficients are given by*

$$a^*_{n,l} = \int_{\partial \mathcal{S}} \overline{\boldsymbol{\mathcal{S}}^*_{n,l}}\, \mathbf{f}\, \boldsymbol{\omega}\, d\sigma \quad (l = 0, \ldots, n)$$

with weight function

$$\boldsymbol{\omega} := |c^2 - (ca\cos\theta + \mathbf{i}\, cb \sin\theta)^2|^{1/2} (\sin^2\theta + \sinh^2\mu)$$

such that $a > b$.

4 MONOGENIC SZEGÖ KERNEL FUNCTION OVER 3D SPHEROIDS

Due to the absence of a direct analogue of the famous Riemann mapping theorem for higher dimensions, at first glance it seems extremely difficult to get closed formulae for the Szegö kernel on monogenic functions. However, in 2002 D. Constales and R. Kraußhar [4] provided an important breakthrough in this research direction. As far as we know, before their work explicit formulae for the Bergman kernels were only known for very special domains, such as for instance the unit ball and the half-space. In several papers [5, 6, 7, 8], the authors were able to give explicit representation formulae for the monogenic Bergman kernel for block domains, wedge shaped domains, cylinders, triangular channels and hyperbolic polyhedron domains which are bounded by parts of spheres and hyperplanes. Recently R. Kraußhar et al. also managed to set up explicit formulae for the Bergman kernel of polynomial Dirac equations, including the Maxwell-, Helmholtz- and Klein-Gordon equations as special subcases, for spheres and annular shaped domains.

With the help of the above-mentioned polynomials we may now obtain an explicit representation for the monogenic Szegö kernel function over 3D prolate spheroids. Now, since the right linear set $\mathcal{M}^+(\mathcal{S}; \mathbb{H})$ is a subspace of $L^2(\mathcal{S}; \mathbb{H}; \mathbb{H})$, to each $\xi \in \mathcal{S}$, if $\mathbf{K}(x, \xi)$ is a positive definite Hermitian quaternion element in $\mathcal{M}^+(\mathcal{S}; \mathbb{H})$, then it can be easily shown that there exists a uniquely determined Hilbert space of functions admitting the reproducing kernel $\mathbf{K}(x, \xi)$, and such that

$$\mathbf{f}(\xi) = \int_{\partial \mathcal{S}} \overline{\mathbf{f}}\, \mathbf{K}(x, \xi)\, \boldsymbol{\omega}\, d\sigma(x),$$

for any $\mathbf{f} \in \mathcal{M}^+(\mathcal{S}; \mathbb{H})$. The function $\mathbf{K}(x, \xi)$, with $(x, \xi) \in \mathcal{S} \times \mathcal{S}$, is called the monogenic Szegö kernel function of $\mathcal{S}$ with respect to ξ, and is given by

$$\mathbf{K}(x, \xi) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \boldsymbol{\mathcal{S}}^*_{n,l} \int_{\partial \mathcal{S}} \overline{\boldsymbol{\mathcal{S}}^*_{n,l}}\, \mathbf{K}(x, \xi)\, \boldsymbol{\omega}\, d\sigma(x).$$

Next we formulate our main result.

Theorem 4.1 (see [32]). *The monogenic Szegö kernel of $\mathcal{S}$*

$$\mathbf{K} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{H}$$

is given explicitly by the formula

$$\mathbf{K}\Big((\mu,\theta,\phi),(\eta,\beta,\varphi)\Big) = \frac{1}{4}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(n+2+l)^2(n+1+l)(n-l+1)}{\|\boldsymbol{\mathcal{S}}_{n,l}\|^2_{L^2(\partial\mathcal{S};\mathbb{H};\mathbb{H})}}\,(A+B+C+D),$$

with the subscript coefficient functions

$$\begin{aligned}
A &= (n+1+l)(n-l+1)A_{n,l}(\mu,\theta)\,A_{n,l}(\eta,\beta)\Big\{\cos[l(\phi-\varphi)] - \sin[l(\phi+\varphi)\mathbf{k}]\Big\},\\
B &= -\frac{A_{n,l+1}(\mu,\theta)\,A_{n,l+1}(\eta,\beta)}{(n+1+l)(n-l+1)}\Big\{\cos[(l+1)(\phi-\varphi)] + \sin[(l+1)(\phi-\varphi)]\mathbf{k}\Big\},\\
C &= A_{n,l}(\mu,\theta)\,A_{n,l+1}(\eta,\beta)\Big\{\cos[l\phi-(l+1)\varphi]\mathbf{i} - \sin[l\phi-(l+1)\varphi]\mathbf{j}\Big\},\\
D &= A_{n,l+1}(\mu,\theta)\,A_{n,l}(\eta,\beta)\Big\{\cos[(l+1)\phi+l\varphi]\mathbf{i} + \sin[(l+1)\phi+l\varphi]\mathbf{j}\Big\}.
\end{aligned}$$

for $l = 0,\dots,n$.

Ultimately, we recall some of the basic properties of $\mathbf{K}$.

Proposition 4.2 (see [32]). *The monogenic Szegö kernel function $\mathbf{K}$ satisfies the following properties:*

1. $\mathbf{K}\Big((0,0,0),(0,0,0)\Big) = \begin{cases} \frac{1}{2}\frac{(n+2)^2(n+1)^4}{\|\boldsymbol{\mathcal{S}}_{n,0}\|^2_{L^2(\partial\mathcal{S};\mathbb{H};\mathbb{H})}}\left|\sum_{k=0}^{\lceil\frac{n}{2}\rceil}\frac{(2n+1-2k)\,(n)_{2k}}{(n+1)_{2k+1}}\right|^2 & l=0\\ 0 & l>0\end{cases}$;
2. *The function $\mathbf{K}$ is 2π-periodic with respect to the variables ϕ and φ.*

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REFERENCES

[1] R. Almeida and H. Malonek. *On a Higher Dimensional Analogue of the Joukowski Transformation*. AIP Conf. Proc. 1048, 630-633 (2008).

[2] J.P. Boyd. *Prolate spheroidal wavefunctions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudospectral algorithms*. J. Comput. Phys., 199(2): 688-716, 2004.

[3] Q.Y. Chen, D. Gottlieb and J.S. Hesthaven. *Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs*. SIAM J. Numer. Anal., 43(5): 1912-1933, (2005).

[4] D. Constales, R. Kraußhar. *Bergman Kernels for rectangular domains and Multiperiodic Functions in Clifford Analysis*. Mathematical Methods in the Applied Sciences 25 No. 16-18, 1509-1526 (2002).

[5] D. Constales and R. Kraußhar. *The Bergman Kernels for the half-ball and for fractional wedge-shaped domains in Clifford Analysis*. Forum Mathematicum 17 (5), 809-821 (2005).

[6] D. Constales and R. Kraußhar. *Hilbert Spaces of Solutions to Polynomial Dirac equations, Fourier Transforms and Reproducing Kernel Functions for Cylindrical Domains*. Zeitschrift fur Analysis und Ihre Anwendungen 24 (3), 611-636 (2005).

[7] D. Constales and R. Kraußhar. *Bergman Spaces of higher dimensional hyperbolic polyhedron type domains I*. Mathematical Methods in the Applied Sciences 29 (1), 85-98 (2006).

[8] D. Constales and R. Kraußhar. *On the Navier-Stokes equation with Free Convection in three dimensional unbounded triangular channels*. Mathematical Methods in the Applied Sciences 31 (6), 735-751 (2008).

[9] C. Cruz, M. I. Falcão, H. Malonek. *3D Mappings by Generalized Joukowski Transformations*. Computational Science and Its Applications - ICCSA 2011, Lecture Notes in Computer Science, Volume 6784/2011, 358-373, Santander (2011).

[10] R. Delanghe. *On homogeneous polynomial solutions of the Riesz system and their harmonic potentials*. Complex Variables and Elliptic Equations, Vol. 52, No. 10-11, 1047–1062 (2007).

[11] R. Delanghe. *On homogeneous polynomial solutions of the Moisil-Théodoresco system in* $\mathbb{R}^3$, Computational Methods and Function Theory, Vol. 9, No. 1, 199–212 (2009).

[12] C. Flammer. *Spheroidal Wave Functions*. Stanford University Press, Stanford, CA, 1957.

[13] R. Fueter. *Analytische Funktionen einer Quaternionenvariablen*. Comment. Math. Helv., 4, 9-20 (1932).

[14] R. Fueter. *Functions of a Hyper Complex Variable*. Lecture notes written and supplemented by E. Bareiss, Math. Inst. Univ. Zürich, Fall Semester, 1949.

[15] P. Garabedian. *Orthogonal harmonic polynomials*. Pacific J. Math. Vol. 3, No. 3, 585-603 (1953).

[16] S. Georgiev and J. Morais. *On convergence aspects of Spheroidal monogenics*. AIP Conference Proceedings, Vol. 1389, 276-279 (2011).

[17] K. Gürlebeck, K. Habetha and W. Sprößig, *Holomorphic Functions in the Plane and n-dimensional Space*, Birkhäuser Verlag, Basel - Boston - Berlin, 2008.

[18] K. Gürlebeck and J. Morais. *On Local Mapping Properties of Monogenic Functions*, K. Gürlebeck and C. Könke (Editors): Proceedings 18th International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering, Weimar (2009).

[19] K. Gürlebeck and J. Morais. *Local properties of monogenic mappings*, AIP Conference Proceedings, "Numerical analysis and applied mathematics", 797-800 (2009).

[20] K. Gürlebeck and J. Morais. *Geometric characterization of M-conformal mappings*, Geometric Algebra Computing: in Engineering and Computer Science, Bayro-Corrochano, Eduardo; Scheuermann, Gerik (Eds.), Springer, 1st Edition, 327-342 (2010).

[21] E. W. Hobson. *The theory of spherical and ellipsoidal harmonics*, Cambridge, 1931.

[22] V. Kravchenko and M. Shapiro. *Integral Representations for Spatial Models of Mathematical Physics*. Research Notes in Mathematics, Pitman Advanced Publishing Program, London, 1996.

[23] K.I. Kou, J. Morais and Y. Zhang. *Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis*. To appear in Mathematical Methods in the Applied Sciences.

[24] N.N. Lebedev. *Special Functions and their Applications*. Dover, New York, Chs. 7, 8, 1972.

[25] H. Leutwiler. *Quaternionic analysis in $\mathbb{R}^3$ versus its hyperbolic modification*. Brackx, F., Chisholm, J.S.R. and Soucek, V. (ed.). NATO Science Series II. Mathematics, Physics and Chemistry, Vol. 25, Kluwer Academic Publishers, Dordrecht, Boston, London (2001).

[26] H. Malonek and M.I Falcão. *3D-mappings by means of monogenic functions and their approximation*. Mathematical Methods in the Applied Sciences, Vol. 33, No. 4, 423-430 (2010).

[27] H. Malonek and R. Almeida. *A note on a generalized Joukowski transformation*. Applied Mathematics Letters, 23, 1174-1178 (2010).

[28] G. Moisil and N. Théodoresco. *Fonctions holomorphes dans l'espace*. Matematica (Cluj) 5, 142-159 (1931).

[29] J. Morais. *A Complete Orthogonal System of Spheroidal Monogenics*. To appear in the Journal of Numerical Analysis, Industrial and Applied Mathematics.

[30] J. Morais and M. Ferreira. *Quasiconformal mappings in 3D by means of monogenic functions*. To appear in Mathematical Methods in the Applied Sciences, 13 pp.

[31] J. Morais. *An orthogonal system of monogenic polynomials over prolate spheroids in* $\mathbb{R}^3$. Submitted for publication.

[32] J. Morais, S. Georgiev, K.I. Kou, T. Qian and W. Sprößig. *Generalized holomorphic Szegö kernel in 3D spheroids*. Submitted for publication.

[33] M. Riesz. *Clifford numbers and spinors*. Institute for Physical Science, Vol. 54, Kluwer Academic Publishers, Dorrecht, 1993.

[34] D. Slepian and H. O. Pollak. *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty, I*, Bell System Tech. J., Vol. 40, 1961, pp. 43-64.

[35] D. Slepian. *Prolate spheroidal wave functions, Fourier analysis and uncertainity. IV. Extensions to many dimensions; generalized prolate spheroidal functions.* Bell System Tech. J., 43:3009-3057, 1964. MR0181766 (31:5993).

[36] E. M. Stein and G. Weiß. *On the theory of harmonic functions of several variables*. Part I: *The theory of* H^p *spaces*. Acta Math. 103, 25-62 (1960).

[37] E.M Stein and G. Weiss. *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. 90, 163-196 (1968).

[38] A. Sudbery. *Quaternionic analysis*. Math. Proc. Cambridge Phil. Soc. 85: 199-225 (1979).

[39] G. Szegö. *A problem concerning orthogonal polynomials*, Trans. Amer. Math. Soc. 37, 196-206 (1935).

[40] G. Szegö. *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, New York (1939).

[41] G. Walter and T. Soleski. *A new friendly method of computing prolate spheroidal wave functions and wavelets.* Appl. Comput. Harmon. Anal., 19(3): 432-443, (2005).

[42] G. Walter. *Prolate spheroidal wavelets: Translation, convolution, and differentiation made easy.* J. Fourier Anal. Appl., 11(1): 73-84, (2005).

[43] G. Walter *Wavelets based on prolate spheroidal wave functions.* J. Fourier Anal. Appl., 10(1): 1-26 (2004).

[44] A.I. Zayed. *A generalization of the prolate spheroidal wave functions*. Proc. Amer. Math. Soc., 135(7): 2193-2203 (electronic), 2007.